HW 02 – Ex. 7

$[\underline{t}_1, \underline{t}_2, \underline{t}_3] \in \mathbb{R}^{3\times3} \equiv T$

$[\underline{n}_1, \underline{n}_2, \underline{n}_3] \in \mathbb{R}^{3\times3} \equiv N$

Q: what is $[\underline{\underline{\sigma}}]$?

$T = \underline{\underline{\sigma}} N$

$\underline{\underline{\sigma}} = T N^{-1}$

7.1 rewrite $\underline{\underline{\sigma}}$ as a vector

$$\tilde{\sigma} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yx} \\ \vdots \\ \sigma_{zz} \end{pmatrix} \in \mathbb{R}^{9\times1}$$

$$\begin{bmatrix} | & | & | \\ t_1 & t_2 & t_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} | & | & | \\ n_1 & n_2 & n_3 \\ | & | & | \end{bmatrix}$$

$A\tilde{\sigma} = b$

$\cancel{A}$

$$\underline{t}_1 = \begin{bmatrix} -n_1- & \overset{(0,0,0)}{\downarrow}\text{———} & \text{———} \\ \text{———} & -n_1- & \text{———} \\ & & \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \\ \vdots \\ \sigma_{zz} \end{pmatrix}$$

$$A = \begin{bmatrix} -n_1- & \text{———} & \text{———} \\ \text{———} & -n_1- & \text{———} \\ \text{———} & \text{———} & n_1 \\ -n_2- & \text{———} & \text{———} \\ \text{———} & -n_2- & \text{———} \\ \text{———} & \text{———} & -n_2- \\ -n_3- & \text{———} & \text{———} \\ \text{———} & n_3 & \text{———} \\ \text{———} & \text{———} & n_3 \end{bmatrix} \in \mathbb{R}^{9\times9}$$

Γ_1

$$b = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \in \mathbb{R}^{3\times 1}$$

$$x = \tilde{\sigma} \in \mathbb{R}^{3\times 1}$$

$\to$ solve $Ax = b$

$\tilde{\sigma}$

solution $\cancel{[\tilde{\sigma}]}$ $[\underline{\sigma}] = \begin{pmatrix} 5 & 3 & -3 \\ 3 & 0 & 2 \\ -3 & 2 & 0 \end{pmatrix}$

<u>7.2</u> $N = (\underline{n_1}, \underline{n_2}, \underline{n_3}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$

rotation of $-45° = \frac{-\pi}{4} =: \alpha$

$$\underline{\underline{R}}^2(\alpha) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{\tilde{T}}} = \underline{\underline{R}}\,\underline{\underline{T}} = \begin{pmatrix} 5.5 & -2.5 & -\frac{1}{\sqrt{2}} \\ -2.5 & -0.5 & 5/\sqrt{2} \\ -\frac{1}{\sqrt{2}} & 5/\sqrt{2} & 0 \end{pmatrix}$$

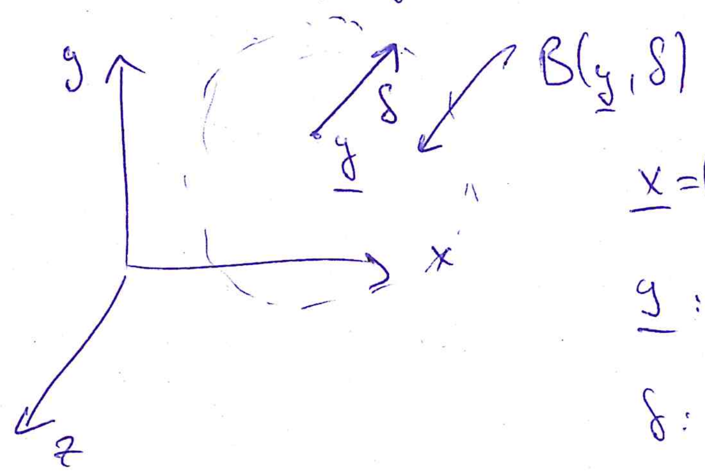
$$\tilde{N} = R\,N = I$$

$R = N^{-1}$ by construction

$$\underline{\underline{T}} = \underline{\underline{\sigma}}\, N$$

$$\underline{\underline{\tilde{T}}} = \underline{\underline{\tilde{\sigma}}}\, I \;\rightarrow\; \underline{\underline{\tilde{\sigma}}} = \underline{\underline{\tilde{T}}}$$

Problem 8 : Divergence Theorem (D.T)



$\underline{x} = (x, y, z)$

$\underline{y}$: center of $B(\underline{y}, \delta)$

δ : radius of B

The D.T. states

$$\int_{\partial B} \underline{v}(\underline{x}) \cdot \underline{n}(\underline{x}) \, dS = \int_{B(\underline{y}, \delta)} \nabla \cdot \underline{v}(\underline{x}) \, dV$$

$$\shortparallel$$

$$\ldots$$

Substitute $f(x) = \nabla \cdot v(x)$.

Now we write the Taylor series of $f(x)$ around $\underline{y}$.

Rotation of $\underline{\underline{\tilde{\sigma}}}$:

$$\underline{\underline{\sigma}} = \underline{\underline{R}}' \, \underline{\underline{\tilde{\sigma}}} \, R'^T$$

R' ?

$$R' = R^2(+45^\circ) = R^T$$

$$\underline{\underline{\sigma}} = R^T \, \underline{\underline{\tilde{\sigma}}} \, R$$
$$= R^T \, \underline{\underline{\tilde{T}}} \, R$$

$$[\underline{\underline{\sigma}}] = (\text{same as in } 7.1)$$

$$f(\underline{x}) = f(\underline{y}) + \nabla f(\underline{x})\Big|_{\underline{x}=\underline{y}} \cdot \underbrace{(\underline{x}-\underline{y})}_{\equiv \Delta \underline{x}} + \mathcal{O}(\Delta \underline{x}^2)$$

Now let's look at $\Delta \underline{x}$ and realize that

$\|\Delta \underline{x}\| \leq \delta$ for all $\underline{x}$ in $B(\underline{y}, \delta)$.

Now take only zeroth order of the Taylor expansion:

$$f(\underline{x}) = f(\underline{y}) + \mathcal{O}(\Delta \underline{x}) = f(\underline{y}) + \mathcal{O}(\delta)$$

Resubstitute : ~~and write out the~~

$$\int_B \nabla \cdot \underline{v}(\underline{x})\, dV = \int_{B(\underline{y},\delta)} (\nabla \cdot v)(\underline{y}) + \mathcal{O}(\delta)\, dV$$

$$= \underbrace{\int_B 1\, dV}_{vol(B)} \cdot \left(\nabla \cdot v(\underline{y}) + \mathcal{O}(\delta) \right)$$

Reorder:

$$(\nabla \cdot v)(\underline{y}) = \frac{1}{\mathrm{vol}(B)} \cdot \int_{\partial B} \underline{v}(\underline{x}) \cdot \underline{n}(x)\, dS + \mathcal{O}(\delta)$$

Taking the limit yields:
$(\delta \to 0)$

$$(\nabla \cdot \underline{v})(\underline{y}) = \lim_{\delta \to 0} \frac{1}{\mathrm{vol}(B)} \int_{\partial B} \underline{v}(x) \cdot \underline{n}(x)\, dS$$

<u>Interpretation</u>:

- $(\nabla \cdot \underline{v})(\underline{y}) \;\hat{=}\;$ fluxes across the boundary of ∂B divided by volume.

- if $\underline{v}$ is a velocity, then the flux $\underline{v} \cdot \underline{n}$ is the <u>net volume rate</u> $\left[\frac{\text{vol.}}{\text{time}}\right]$ at which a fluid exits B.

- $(\nabla \cdot \underline{v})(\underline{y})$ is a measure of volume expansion at $\underline{y}$

8.2

Let $\underline{a}$ be a <u>constant</u>, <u>arbitrary</u> vector.

Then

$$\underline{a} \cdot \int_{\Omega} \nabla \cdot \underline{\underline{S}} \, d\Omega = \underline{a} \cdot \int_{\partial\Omega} \underline{\underline{S}} \cdot \underline{n} \, dA$$

$$= \int_{\partial\Omega} \underline{a} \cdot (\underline{\underline{S}} \cdot \underline{n}) \, dA$$

$$= \int_{\partial\Omega} \underbrace{(\underline{\underline{S}}^T \underline{a})}_{\equiv \underline{b}} \cdot \underline{n} \, dA \qquad \underline{b} \text{ is a vector field!}$$

$$\overset{(D.T)}{=} \int_{\Omega} \nabla \cdot \underline{b} \, d\Omega$$

Resub:

$$= \int_{\Omega} \nabla \cdot (\underline{\underline{S}}^T \underline{a}) \, d\Omega$$

$$= \int_{\Omega} \partial_{x_j} (S_{ij} a_i) \, d\Omega$$

$$= a_i \int_{\Omega} \partial_{x_j} S_{ij} \, d\Omega$$

$$= \underline{a} \int_{\Omega} \nabla \cdot \underline{\underline{S}} \, d\Omega$$

~~Since a~~

What we have

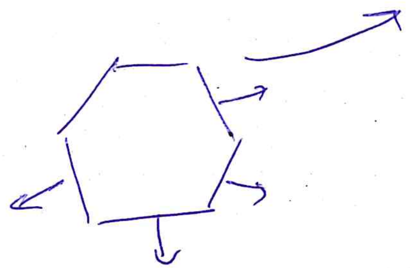
$$a \cdot \int_{\partial\Omega} \underline{\underline{S}} \cdot \underline{n} \, dA = a \cdot \int_{\Omega} \nabla \cdot \underline{\underline{S}} \, d\Omega$$

Since a was arbitrary, we get

$$\boxed{\int_{\partial\Omega} \underline{\underline{S}} \cdot \underline{n} \, dA = \int_{\Omega} \nabla \cdot \underline{\underline{S}} \, d\Omega}$$

$\square$

The FVM in N-D for unstructured grids on simplex meshes.



PDE $\frac{\partial}{\partial t}\underline{Q} + \nabla \cdot \underline{\underline{F}}(\underline{Q}) = \underline{S}$

$\underline{Q} \in \Omega_Q \subset \mathbb{R}^m$

in 2D:

$$\frac{\partial \underline{Q}}{\partial t} + \underbrace{\boxed{\partial_x \underline{f}(\underline{Q})}}_{1D} + \partial_y \underline{g}(\underline{Q}) = \underline{S}$$

$$\underline{\underline{F}} = [\underline{f}, \underline{g}]$$

IC: $\underline{Q}(x, t=0) = \underline{Q}_0(x)$

BC's: appropriate

__Example__ : SWE in 2D

$$\partial_t h + \partial_x (hu) + \partial_y (hv) = 0$$

$$\partial_t (hu) + \partial_x (hu^2 + gh^2/2) + \partial_y (huv) = 0$$

$$\partial_t (hv) + \partial_x (huv) + \partial_y (hv^2 + gh^2/2) = 0$$

$$\underline{Q} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} \quad \underline{f} = \begin{pmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{pmatrix} \quad \underline{g} = \begin{pmatrix} hv \\ huv \\ hv^2 + gh^2/2 \end{pmatrix}$$

$$\underline{\underline{E}} = [\underline{f}, \underline{g}]$$

Given a space time control volume

$\Omega \times T$ (1D: $[x_{i-1/2}, x_{i+1/2}] \times [t^n, t^{n+1}]$)

We now derive the weak integral form of the PDE:

$$\int_{\Omega(\underline{x})} \int_{t^n}^{t^{n+1}} \partial_t \underline{Q} + \nabla \cdot \underline{\underline{F}}(\underline{Q}(t,\underline{x}))\, dt\, dx = 0$$

$$\overset{(D.T)}{\Longleftrightarrow} \int_{\Omega} \underline{Q}(t^{n+1},\underline{x}) - \underline{Q}(t^n,\underline{x})\, dx + \int_{t^n}^{t^{n+1}} \oint_{\partial\Omega} \underline{\underline{F}}(\underline{Q}(t,\underline{x})) \cdot \underline{n}\, dt = 0$$

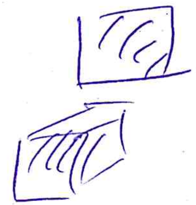
<u>Define</u>:

$$\underline{Q}_i^n = \frac{1}{|\Omega_i|} \int_{\Omega} \underline{Q}(t^n,\underline{x})\, dx$$

1D : dx

2D : area

3D :

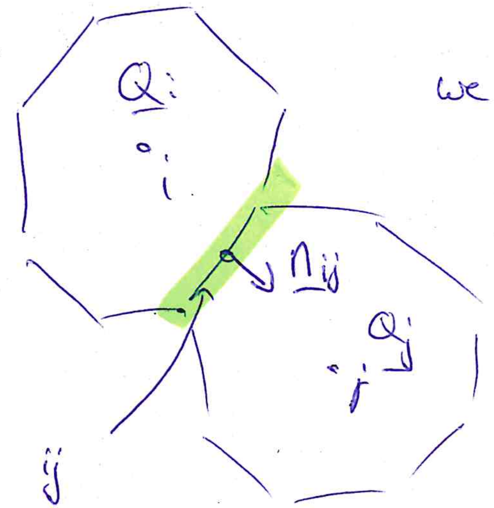


$$\tilde{\underline{\underline{F}}} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \underline{\underline{F}}(Q(t, \tilde{x})) \, dt$$

$$\left(1D: \quad F_{i+1/2} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(Q(t, x_{i+1/2})) \, dt \right)$$

$\tilde{\underline{\underline{F}}}$ denotes $\underline{\underline{F}}$ at the interface $\tilde{x}$.

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{|\Omega_i|} \oint_{\partial\Omega} \tilde{\underline{\underline{F}}} \cdot \tilde{n} \, dA \quad \Rightarrow$$



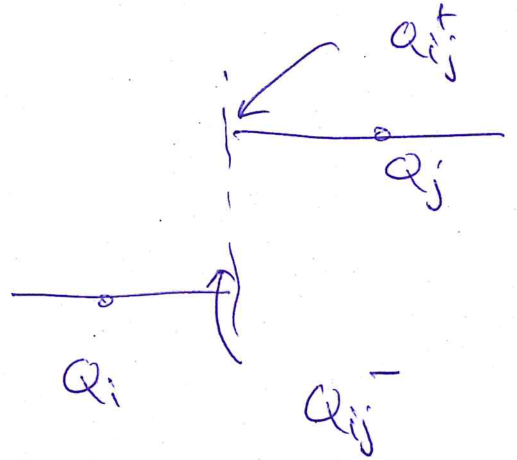
we assume that n_{ij} is pointing from $\underline{x}_i$ to $\underline{x}_j$.

TR

Now

$$\underline{\underline{\tilde{F}}} = \underline{\underline{F}}_{ij} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \underline{\underline{F}}(Q(t, x_{ij})) \, dt$$

At the interface ; we have



⇝ Reconstruction problem.

For constant recon st.

$$Q_{ij}^- = Q_i$$

$$Q_{ij}^+ = Q_j$$

Lax Friedrich in 1D:

$$\underline{f}_{i+1/2} = \underline{f}_{ij}^{1D} = \frac{1}{2}\left(\underline{f}_i + \underline{f}_j\right) - \frac{\Delta x}{2\Delta t}\left(\underline{Q}_j - \underline{Q}_i\right)$$

$$f_i \equiv f(Q_i)$$

in 2D:

$$\underline{\underline{f}}_{ij} \cdot \underline{n}_{ij} \cancel{= \frac{1}{2}\left(\underline{\underline{F}}_{ij} \cdot n_{ij}\right.}$$

$$= \frac{1}{2}\left(\underline{\underline{F}}(Q_i) \cdot n_{ij} + F(Q_j) \cdot n_{ij}\right)$$

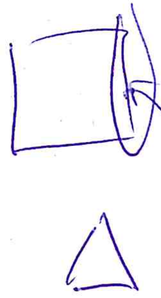
$$- \frac{d^{min}}{2\Delta t}\left(Q_j - Q_i\right)$$

d^{min}: minimal inradius of all cells

Resoluing: $\oint_{\partial\Omega} \underline{\underline{F}}(Q(t,\tilde{x})) \cdot \tilde{n} \, dA$

~~Simplex meshes~~

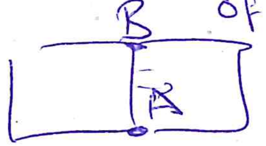
with simplex meshes, where a cell i han 'n_neighbors' of neighboring cells



n_neighbor = 4

3

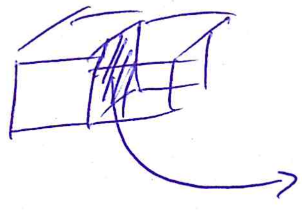
$|l_{ij}|$: length/area of the interface



$|l_{ij}| = |\overline{BA}|$

lengh : 2D

area : 3D



$|l_{ij}|$

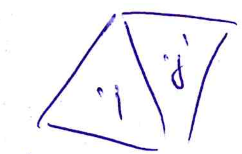
Now we can write

$$\oint_{\partial\Omega} \tilde{\underline{\underline{F}}} \cdot \tilde{n} = \sum_{j=1}^{n\_neighbors} \underline{\underline{F}}_{ij} \cdot \underline{n}_{ij} \cdot |\Omega_{ij}|$$

$$\boxed{Q_i^{n+1} = Q_i^n - \frac{\Delta t}{|\Omega_i|} \sum_{j=1}^{n\_neighbors} \underline{\underline{F}}_{ij} \cdot \underline{n}_{ij} \, |\Omega_{ij}|}$$

<u>Ingredients</u>

<u>Geometry</u>: cells + neighbors



$|\Omega_i|$: volume of cell

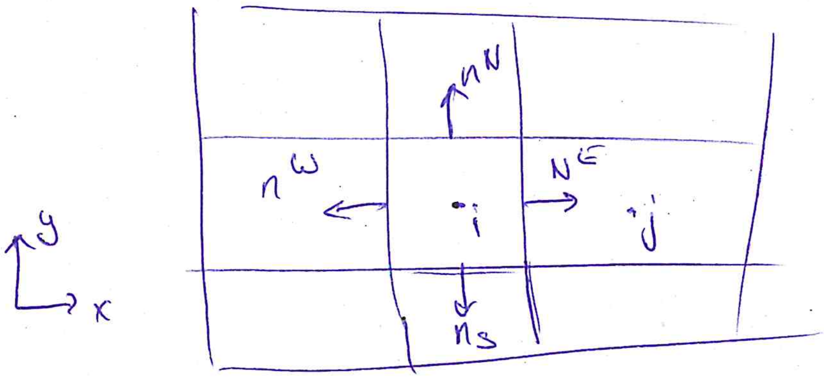
$|\Omega_{ij}|$: volume of the interface

$\underline{n}_{ij}$: outward facing from i

d^{min} : minimal incircle

- $\underline{\underline{F}}$: flux tensor $\to$ PDE
- Δt $\to$ constant (CFL)
- Q_i^0 : IC
- Riemann solver.

<u>Special case</u>: 2D uniform cartesian mesh.



$|\Omega_i| = \text{const.}$

$|\Omega_{ij}| = \text{const.}$

n_{ij} : $n^N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $n^S = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $n^O = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $n^E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

TN

Now look at

$$\underline{\underline{F}}_{ij} \cdot n_{ij} = [\underline{f}, \underline{g}] \cdot n_{ij}$$

$$\underline{\underline{F}}_{ij} \cdot n^E = \underline{\underline{F}}_{ij} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{f}_{ij}^E$$

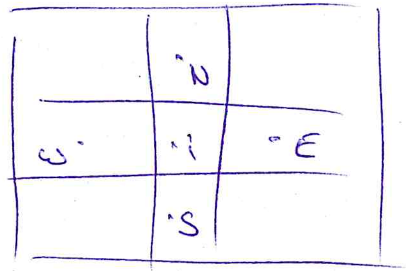
$$\underline{\underline{F}}_{ij} \cdot n^N = \underline{\underline{F}}_{ij} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = g_{ij}^N$$

Then

$$\sum_{j=1}^{4} \underline{\underline{F}}_{ij}\, \underline{n}_{ij} \cdot |l_{ij}| = |l_{ij}| \cdot \left(\underline{f}_{iE} - \underline{f}_{iW} + \underline{g}_{iN} - \underline{g}_{iS} \right)$$

$$\partial_t Q + \partial_x \underline{f}(Q) + \partial_y \underline{g}(Q) = 0$$

$$Q_i^{n+1} = Q_i^n + \frac{\Delta t}{\cancel{\Delta x}} \underbrace{\frac{|l_{ij}|}{|l_i|}}_{= \text{const}} \cdot \left(\underline{f}_{iE} - \underline{f}_{iW} + \underline{g}_{iN} - \underline{g}_{iS} \right)$$

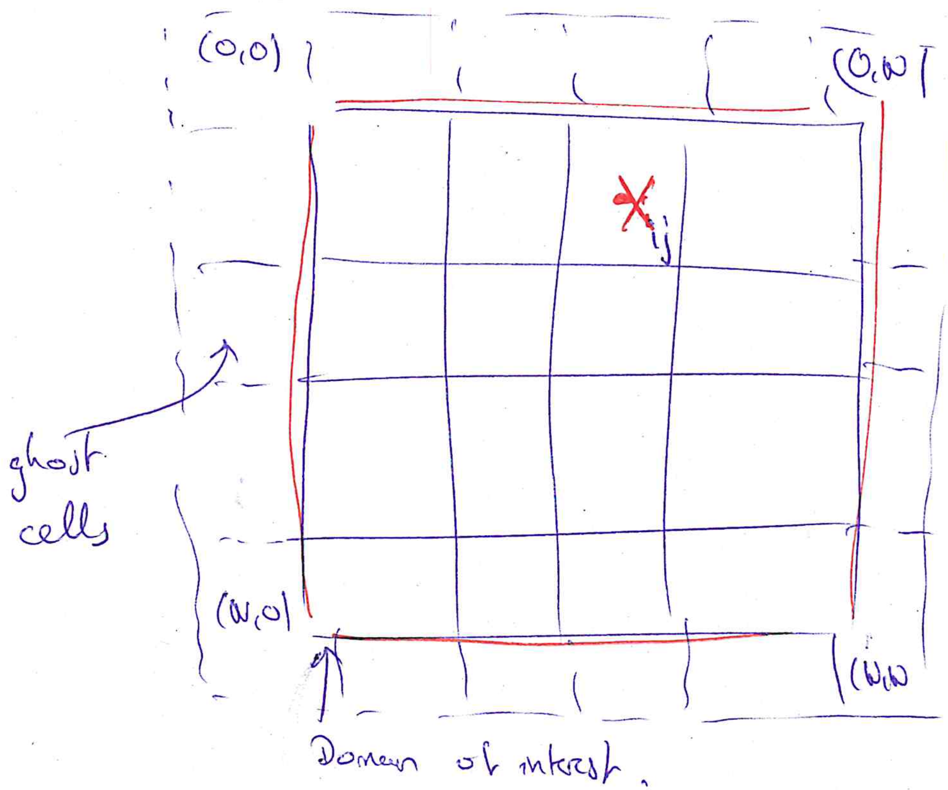


$$f_{iE} = f^{LF}(Q_i, Q_E, d^{min}, \Delta t, \cancel{f_{iE}})$$

$$= \frac{1}{2}\left(f(Q_i) + f(Q_E)\right) - \frac{d^{min}}{2\Delta t}(Q_E - Q_i)$$

$$g_{iE} = g^{LF} = \frac{1}{2}\left(g(Q_i) + g(Q_N)\right) - \frac{d^{min}}{2\Delta t}(Q_N - Q_i)$$

Coding :



$$Q \in \mathbb{R}^{(dof \times N \times N)}$$

$\uparrow$ 3 for SWE, 2D

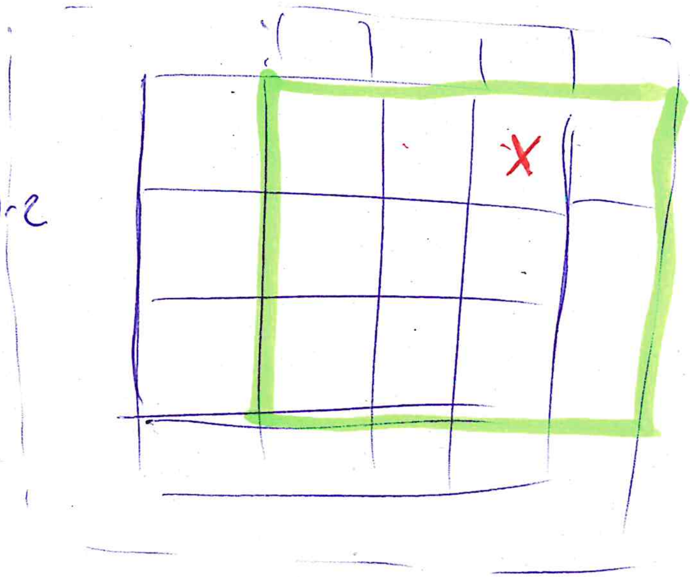
$$Q^{inner} = Q[:,\ 1:-1\ ,\ 1:-1] \quad \cancel{\equiv Q_i \in \mathbb{R}^{dof \times N-2 \times N-2}}$$

$1 : N-1$

$Q^{inner} \hat{=} Q_i \in \mathbb{R}^{dof \times N-2 \times N-2}$

Q_E:

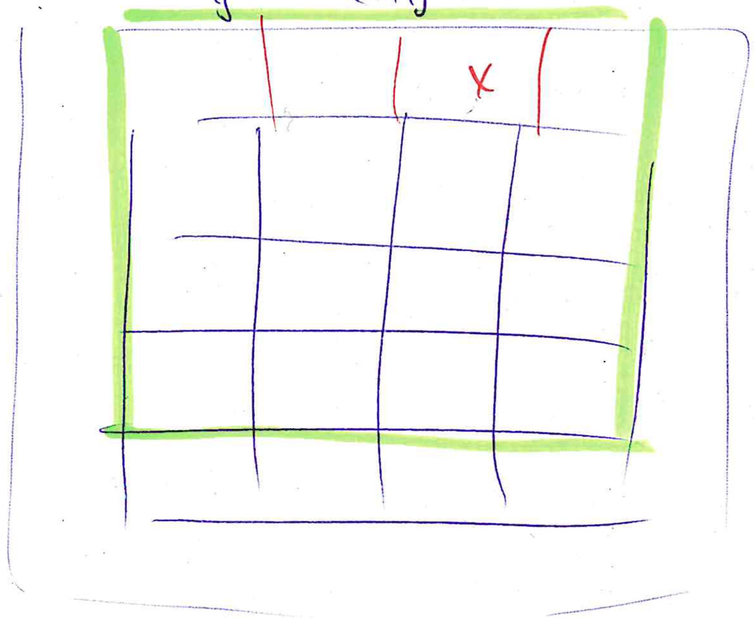
$\in \mathbb{R}^{dof \times N-2 \times N-2}$



Ex:

$Q^{inner} + Q^E = Q_{ij} + Q_{(i+1)j}$

Q_N:



Ex:

$Q^{inner} + Q^W = Q_{ij} + Q_{i(j+1)}$